Computational Models - Lecture 6[1]

Handout Mode

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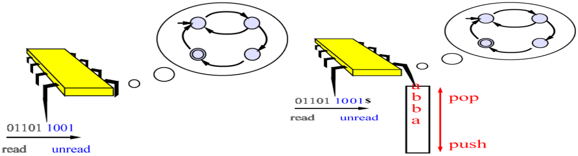
[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University. Also with modifications of Yishay Mansour.

Outline

- Push Down Automata (PDA)
- Closure properties for CFL and testing properties.
- Equivalence of CFGs and PDAs



- Sipser's book, 2.1, 2.2 & 2.3

Part I

Push-Down Automata

Diagram Notation

When drawing the automata diagram, we use the following notation

- Transition from state q to state q' labelled by $a, b \to c$ means $(q', c) \in \delta(q, a, b)$,
  and informally means the automata
  - read a from input
  - pop b from stack
  - push c onto stack
- Meaning of ε transitions ((informally):
  - $a = \varepsilon$: don't read input
  - $b = \varepsilon$: don't pop any symbol
  - $c = \varepsilon$: don't push any symbol

How to define $\widehat{\delta}(q, w, s)$ contains (q', s')?

Given (start) state q, substring w of the input, and s, s' descriptions of strings on a stack:
There is a legal way to get from state q with stack contents s to state q' with stack contents s' by reading from w at each step.

Model of Computation

The following is with respect to $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$.

Definition 1 (δ^*)

For $w \in \Sigma^*$ let $\widehat{\delta}(q, w, s)$ be all pairs $(q', s') \in Q \times \Gamma^*$ for which exist $w'_1, \ldots, w'_m \in \Sigma_\varepsilon$, states $r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots s_m \in \Gamma^*$ s.t.:

1. $w = w'_1, \ldots, w'_m$, $r_0 = q$, $r_m = q'$, $s_0 = s$ and $s_m = s'$
2. For every $i \in \{0, \ldots, m-1\}$ exist $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$ s.t.:
   2.1 $(r_{i+1}, b) \in \delta(r_i, w'_{i+1}, a)$
   2.2 $s_i = at$ and $s_{i+1} = bt$

Namely, $(q', s') \in \widehat{\delta}(q_0, w, \varepsilon)$ if after reading w (possibly with in-between ε moves), M can find itself in state q' and stack value s'.

- M accepts $w \in \Sigma^*$ if $\exists q' \in \mathcal{F}$ such that $(q', t) \in \widehat{\delta}(q_0, w, \varepsilon)$ for some t.

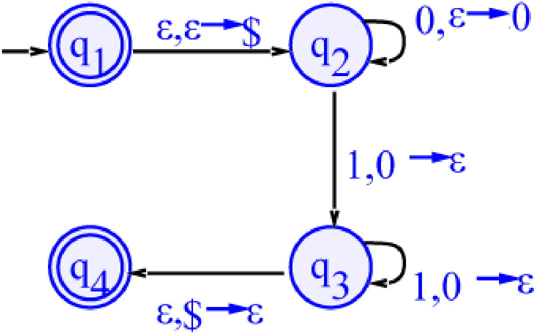
Knowing when stack is empty

It is convenient to be able to know when the stack is empty, but there is no built-in mechanism to do that.

Solution

1. Start by pushing $\$$ onto stack.
2. When you see it again, stack is empty.

A PDA for $\mathcal{L}_1 = \{0^n 1^n \colon n \geq 0\}$



Claim 2

$0011 \in L(P)$.

Proof: For input read bits w', stack contents s_i, current state r_i, take

	$w'_1 = \varepsilon$	$w'_2 = 0$	$w'_3 = 0$	$w'_4 = 1$	$w'_5 = 1$	$w'_6 = \varepsilon$
$s_0 = \varepsilon$	$s_1 = \$$	$s_2 = 0\$$	$s_3 = 00\$$	$s_4 = 0\$$	$s_5 = \$$	$s_6 = \varepsilon$
$r_0 = q_1$	$r_1 = q_2$	$r_2 = q_2$	$r_3 = q_2$	$r_4 = q_3$	$r_5 = q_3$	$r_6 = q_4$

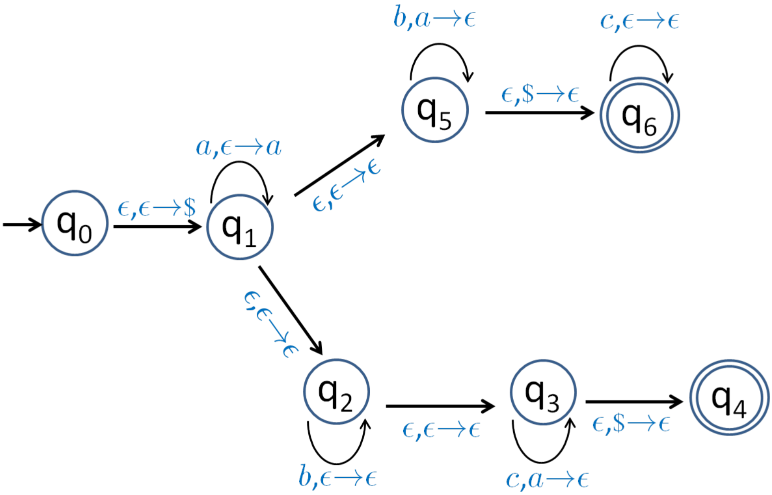
A PDA for $\mathcal{L}_1 = \{0^n 1^n \colon n \geq 0\}$

We want to show that $L(P) = \mathcal{L}_1 = \{0^n 1^n \colon n \geq 0\}$
What do we need to prove?

Claim 3

- $\widehat{\delta}(q_1, \varepsilon, \varepsilon) = \{(q_1, \varepsilon), (q_2, \$)\}$.
- $\widehat{\delta}(q_1, 0^k, \varepsilon) = \{(q_2, 0^k \$)\}$, for $k \geq 1$.
- $\widehat{\delta}(q_1, 0^k 1^i, \varepsilon) = \{(q_3, 0^{k-i} \$)\}$, for $k > i \geq 1$.
- $\widehat{\delta}(q_1, 0^k 1^k, \varepsilon) = \{(q_3, \$), (q_4, \varepsilon)\}$, for $k \geq 1$.
- $\widehat{\delta}(q_1, w, \varepsilon) = \emptyset$, for $w \notin \{0^k 1^i | k \geq i \geq 0\}$.

A PDA for $\mathcal{L}_2 = \{a^i b^j c^k : i = j \vee i = k\}$



A PDA for $\mathcal{L}_2 = \{a^i b^j c^k : i = j \vee i = k\}$, cont.

- Non-determinism is essential here!
- Unlike finite automata, non-determinism does add power.
- But we saw deterministic algorithm to decide membership in any CFL (and as we see later, CFLs are exactly the languages decided by PDAs)!
- How to prove that non-determinism adds power?

  ⋮

- Does not seem trivial or immediate.
- Another example: $\mathcal{L} = \{x^n y^n : n \geq 0\} \cup \{x^n y^{2n} : n \geq 0\}$ is accepted by a non-deterministic PDA, but not by a deterministic one. (Proof? Book!)

PDA Languages

The Push-Down Automata Languages, $\mathcal{L}_{\text{PDA}}$, is the set of all languages that can be described by some PDA:

- $\mathcal{L}_{\text{PDA}} = \{\mathcal{L}(M) \colon M \text{ is a PDA}\}$

It is immediate that $\mathcal{L}_{\text{PDA}} \supsetneq \mathcal{L}_{\text{DFA}}$: every DFA is just a PDA that ignores the stack.

- $\mathcal{L}_{\text{CFG}} \subseteq \mathcal{L}_{\text{PDA}}$?
- $\mathcal{L}_{\text{PDA}} \subseteq \mathcal{L}_{\text{CFG}}$?
- $\mathcal{L}_{\text{PDA}} = \mathcal{L}_{\text{CFG}}$!!!

Proof in last hour of class.

Part II

Closure Properties

Simple Closure Properties of Context-Free Languages

- CFL's are closed under
  - Union: $S \to S_1 \mid S_2$
  - Concatenation: $S \to S_1 S_2$
  - Star: $S_{new} \to \varepsilon \mid S_{old} \mid S_{old} S_{new}$
- What about complement and intersection?

Intersection

$$S_1 \to A_1 B_1 \qquad S_2 \to A_2 B_2$$
$$A_1 \to 0A_1 1|\varepsilon \qquad A_2 \to 0A_2|\varepsilon$$
$$B_1 \to 2B_1|\varepsilon \qquad B_2 \to 1B_2 2|\varepsilon$$

$$\mathcal{L}_1 = 0^n 1^n 2^* \qquad \mathcal{L}_2 = 0^* 1^n 2^n$$

- $\mathcal{L}_1 \cap \mathcal{L}_2 = 0^n 1^n 2^n$
- $\mathcal{L}_1$ and $\mathcal{L}_2$ are CFLs (why?),
- But $\mathcal{L}_1 \cap \mathcal{L}_2$ is not a CFL.
- But can't we run two PDA's in parallel, and accept iff both accept??
- What about intersection of a CFL with a regular language?

When CFL Intersects Regular Language

- Are the context free languages closed under intersection with a regular language?
- That is, if $\mathcal{L}_1$ is context free languages, and $\mathcal{L}_2$ is regular, must $\mathcal{L}_1 \cap \mathcal{L}_2$ be context free languages?
- YES!
  - Run PDA $\mathcal{L}_1$ and DFA $\mathcal{L}_2$ “in parallel” (just like the intersection of two regular languages).
  - Formal details omitted (but you should be able to figure them out).

Applications

- Is $\mathcal{L} = \{w \in \{0,1,2\}^* : \#_0(w) = \#_1(w) = \#_2(w)\}$ context free?
  - $\mathcal{L} \cap 0^*1^*2^* = \{0^n1^n2^n : n \geq 0\}$ is not context free.
  - $0^*1^*2^*$ is regular.
  - Context free languages intersected with a regular languages are context free.
  - So $\mathcal{L}$ is not a context free language
- This could also be established using pumping lemma, but proof above is more elegant.

Complementation

The fact that CFLs are not closed under intersection, but are closed under union, implies they are not closed under complementation, as $\mathcal{L}_1 \cap \mathcal{L}_2 = \overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$.

We give a simple example where $\mathcal{L}$ is not CFL but $\overline{\mathcal{L}}$ is.

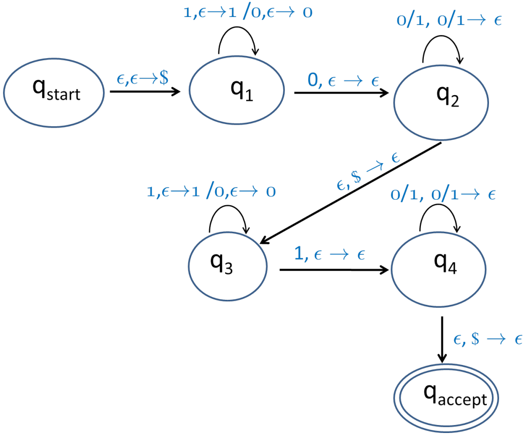
- Take $\mathcal{L} = \{ww : w \in \{0,1\}^*\}$.
- $\mathcal{L}$ is not a CFL (why?)
- We prove that $\overline{\mathcal{L}}$ is a CFL

Complementation cont.

- For any $y \in \overline{\mathcal{L}}$, either
  - y's length is odd, or
  - y's length is even, 2ℓ, and $\exists i \geq 1$ such that $y_i \neq y_{\ell+i}$.
- It suffice to construct a PDA/CFG for $\overline{\mathcal{L}}_{even}$ – the even length members of $\overline{\mathcal{L}}$ (why?)
- Let $\overline{\mathcal{L}}^{\sigma}_{even} = \{\{0,1\}^k \sigma \{0,1\}^j \{0,1\}^k \overline{\sigma} \{0,1\}^j : k, j \geq 0\}$
- Note that $\overline{\mathcal{L}}_{even} = \overline{\mathcal{L}}^{0}_{even} \cup \overline{\mathcal{L}}^{1}_{even}$
- and that $\overline{\mathcal{L}}^{\sigma}_{even} = \{\{0,1\}^k \sigma \{0,1\}^k \{0,1\}^j \overline{\sigma} \{0,1\}^j : k, j \geq 0\}$
- CFG for $\overline{\mathcal{L}}^{0}_{even}$
  $S \to AB$
  $A \to CAC \mid 0$
  $B \to CBC \mid 1$
  $C \to 0 \mid 1$

A PDA for $\overline{\mathcal{L}}^0_{even}$

Idea: Guess $k, j \geq 0$, and accept w if it is of the form:
$\{0,1\}^k 0 \{0,1\}^k \{0,1\}^j 1 \{0,1\}^j$



Homomorphism and Inverse Homomorphism

- *Homomorphism:* replaces each letter with a word
- Example: $h(1) = aba$, $h(0) = aa$
  $h(010) = aa\,aba\,aa$
  $\mathcal{L}_1 = \{0^n1^n | n \geq 0\}$, $h(\mathcal{L}_1) = \{a^{2n}(aba)^n | n \geq 0\}$.
- Claim: Assuming that $\mathcal{L}$ is a CFL, then so is $h(\mathcal{L})$
- Proof? use the grammar
- *Inverse homomorphism:* $h^{-1}(w) = \{x \colon h(x) = w\}$,
  $h^{-1}(\mathcal{L}) = \{x \colon h(x) \in \mathcal{L}\}$
- Example: $\mathcal{L}_2 = \{a^nb^na^i | n, i \geq 0\}$, $h^{-1}(\mathcal{L}_2) = \{10^i, 0^i | i \geq 0\}$.
- Claim: Assuming that $\mathcal{L}$ is a CFL, then so is $h^{-1}(\mathcal{L})$
- Proof? use the automata (see next)

CFL's are Closed Under Inverse Homomorphism

- Idea: let P be a PDA for $\mathcal{L}$. On input word w, emulate $P(h(w))$
- But we cannot afford to store $h(w)$!
- Solution, compute $h(w)$ "on demand":

Algorithm 4

Input w:

1. Initialized a "buffer" Buff to $h(a)$, where a is the first letter of w
2. Emulate a running of P with Buff as its input string.
   Each time Buff is fully read by P, set Buff $= h(a)$, where a is the next letter in w (if exists)
3. `Accept` iff P does

- How do we implement Buff?

CFL's are Closed Under Inverse Homomorphism

Given PDA $(Q, \Sigma, \Gamma, \delta, q_0, F)$ and homomorphism $h : \Sigma_1 \rightarrow \Sigma^*$, we define a new PDA $(Q', \Sigma_1, \Gamma, \delta', q_0', F')$.

- Let $k = \max_{\sigma_1 \in \Sigma_1} |h(\sigma_1)|$ and $\bar{\Sigma} = \cup_{0 \leq i \leq k} \Sigma^i$.
- $Q' = Q \times \bar{\Sigma}$.
- $q_0' = [q_0, \varepsilon]$.
- $F' = F \times \{\varepsilon\}$.
- δ' is define as follows:
  - if $(p, \gamma) \in \delta(q, \varepsilon, Y)$ then $([p, x], \gamma) \in \delta'([q, x], \varepsilon, Y)$, for any $x \in \bar{\Sigma}$.
  - if $(p, \gamma) \in \delta(q, a, Y)$ then $([p, x], \gamma) \in \delta'([q, ax], \varepsilon, Y)$, for any $x \in \bar{\Sigma}$.
  - $([q, h(a)], Y) \in \delta'([q, \varepsilon], a, Y)$, for $a \in \Sigma_1$ and $Y \in \Gamma$.

Part III

Algorithmic Questions

Emptiness of CFGs

Question 5

Given a CFG, G, is $\mathcal{L}(G) = \emptyset$?

In other words, is there a string generated by G?

Theorem 6

There is an algorithm that solves this problem (and always halts).

Possible approaches for a proof:

- Bad Idea: We know how to test whether $w \in \mathcal{L}(G)$ for any string w, so just try it for each w...
- Better Idea: Can the start variable generate a string of terminals?
- A more holistic approach: Can a particular variable generate a string of terminals?

Checking Emptiness

Idea: Mark variables that can produce a string of terminals

1. Mark all terminal symbols in G.
2. Repeat until no new variable become marked:

   Mark any A where $A \rightarrow U_1 U_2 \ldots U_k$ and all U_i have already been marked.
3. Remove all unmarked variables, and any rule they appear in.
4. If S is removed, then $\mathcal{L}(G) = \emptyset$.

Correctness?

CFGs Fullness

Question 7

Given a CFG G, is $\mathcal{L}(G) = \Sigma^*$?

- We just saw an algorithm to determine, given a CFG G, whether $\mathcal{L}(G) = \emptyset$
- $\mathcal{L}(G) = \Sigma^*$ iff $\overline{\mathcal{L}(G)} = \emptyset$. Why not modify the algorithm so it determines emptiness of the complement?
- Unfortunately, CFGs are not closed under complement.

Fact 8

There is no algorithm to solve CFG fullness.

- We are not prepared to prove this remarkable fact (yet).

Finiteness of CFGs

Question 9

Given a CFG G, is $|\mathcal{L}(G)|$ finite?

First, a useful subroutine.

Removing redundant variables and terminals

1. Mark all terminal symbols in G.
2. Repeat until no new variable become marked:
   Mark any A where $A \to U_1 U_2 \ldots U_k$ and all U_i have already been marked.
3. Remove all unmarked variables, and any rule they appear in.
4. If S is removed, then $\mathcal{L}(G) = \emptyset$.
5. Remove any variable A not reachable from S.
6. Remove any terminal which does not appear in some rule.

Back to finiteness of CFGs

Question 10

Given a CFG G, is $|\mathcal{L}(G)|$ finite?

1. Remove redundant variables and terminals.
2. Turn into a CNF form
3. Create a graph C whose nodes are variables and its directed edges are derivations.
4. Return `TRUE` iff C has a no cycles.

Correctness?

CFGs Inherent Ambiguity

Question 11

Given a CFG G, is $\mathcal{L}(G)$ inherently ambiguous?

This means that for any CFG that generates $\mathcal{L}(G)$, there is a word in the language with two different parse trees.

Fact 12

There is no algorithm to solve CFG inherent ambiguity.

- We will not prove this fact, yet you want to know it to put things in context.

When Are Two CFGs equivalent?

Question 13

Given two CFG G_1 and G_2, test if $L(G_1) = L(G_2)$.
Is there an algorithm to solve this problem?

Part IV

Equivalence Theorem

The CFG–PDA Equivalence Theorem

Theorem 14

$\mathcal{L}_{\text{PDA}} = \mathcal{L}_{\text{CFG}}$ *: A language is context free if and only if some pushdown automata accepts it.*

This time (unlike the regular expression vs. regular languages theorem), both the proof "if" part and of the "only if" part are non trivial.

Proof sketch follows.

CFL $\Longrightarrow$ PDA

Lemma 15

$\mathcal{L}_{\text{CFG}} \subseteq \mathcal{L}_{\text{PDA}}$ *: If a language is context free, then some PDA accepts it.*

- Let $\mathcal{L}$ be a context-free language, and let $G = (V, \Sigma, R, S)$ be context-free grammar for $\mathcal{L}$
- We build a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, such that on input w it "figures out" if there is a derivation of w using G.

Question 16

How does P figure out which substitution to make?

Answer: It guesses.

Simplifying Assumptions

1. In a single move, a PDA can push a whole word (from some fixed set) into the stack (first letter at the top)

   Can we justify it?

2. When deriving a word from a CFL, we always substitute the left most variable

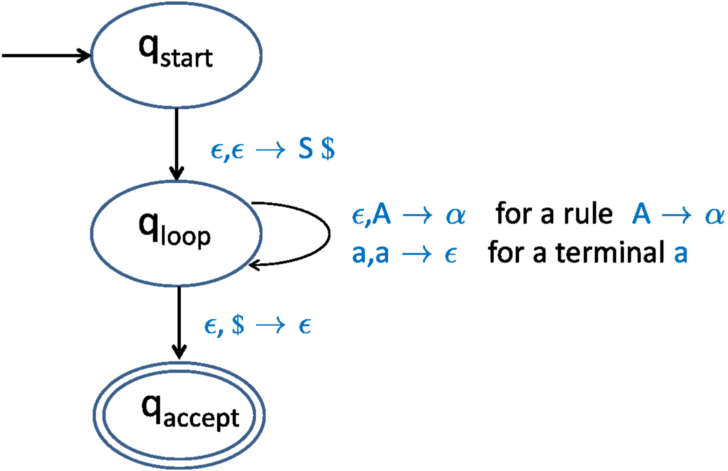
   Does it change the derived language?

Informal Description of P

Algorithm 17 (P)

1. Push $S\$$ on stack
2. While top of the stack t is not $\$$:
3. If t is variable A,
   (non-deterministically) select rule $A \rightarrow \alpha$ and substitute.
4. If t is a terminal a,
   read next input and compare; `Reject` if different.
5. `Accept` if end of input and stack is empty

State Diagram for P

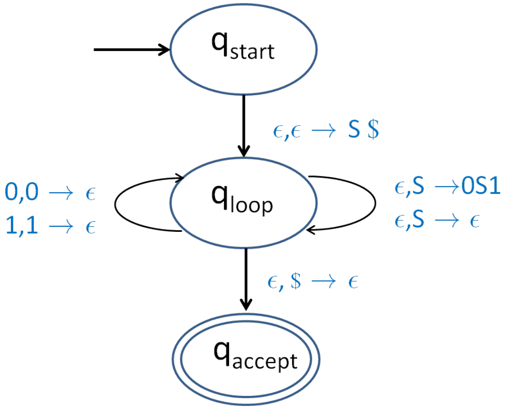


Example

consider the CFG:

$S \to 0S1|\varepsilon$.

The related PDA:



Claim: $\mathcal{L}(P) = \mathcal{L}(G)$

Claim 18

$S \xrightarrow{*} \alpha$ iff $\alpha = \alpha_1\alpha_2$ such that $(q_{loop}, \alpha_2\$) \in \widehat{\delta}(q_{loop}, \alpha_1, \$)$.

Does the above yields that $\mathcal{L}(P) = \mathcal{L}(G)$?

$S \xrightarrow{*} \alpha \implies \alpha = \alpha_1\alpha_2$ such that $(q_{loop}, \alpha_2\$) \in \widehat{\delta}(q_{loop}, \alpha_1, S\$)$

Proof by induction on the number of derivations steps used to yield α from S.

- 1 derivation steps: hence there is a rule $S \to \alpha$. Thus $(q_{loop}, \alpha\$) \in \widehat{\delta}(q_{loop}, \varepsilon, S\$)$, and the proof follows for $\alpha_1 = \varepsilon$ and $\alpha_2 = \alpha$.
- Assume $S \xrightarrow{*} \alpha$ in $k > 1$ derivation steps, and let α' be the string derived by the first $(k-1)$ steps.
- By i.h $\alpha' = \alpha'_1\alpha'_2$ such that $(q_{loop}, \alpha'_2\$) \in \widehat{\delta}(q_{loop}, \alpha'_1, S\$)$
- Write $\alpha'_2 = w_1 A w_2$ where A is the left most variable in α'_2.
- The k'th derivation step replaces this occurrence of A with a string t (why?)
- It is easy to see that $(q_{loop}, tw_2\$) \in \widehat{\delta}(q_{loop}, \alpha'_1 w_1, S\$)$.
- To complete the proof take $\alpha_1 = \alpha'_1 w_1$ and $\alpha_2 = tw_2$.

$\alpha = \alpha_1\alpha_2$ such that $(q_{loop}, \alpha_2\$) \in \widehat{\delta}(q_{loop}, \alpha_1, \$) \implies S \xrightarrow{*} \alpha$

Proof by induction on the number of steps used by P to process α_1.

- A single step: $\alpha_1 = \varepsilon$ and $\alpha_2 = S\$$, and the proof follows since $S \xrightarrow{*} S$.
- Assume α_1 was processed in $k > 1$ steps, and let α'_1 and α'_2 be the input string read and the stack value *before* the last step
- Note that $(q_{loop}, \alpha'_2\$) \in \widehat{\delta}(q_{loop}, \alpha'_1, \$)$.
- By i.h $S \xrightarrow{*} \alpha' = \alpha'_1\alpha'_2$.
- In case the k'th move of P is reading an input character, then $\alpha_1\alpha_2 = \alpha'_1\alpha'_2$, and therefore $S \xrightarrow{*} \alpha_1\alpha_2$
- Otherwise, $\alpha'_1 = \alpha_1$, $\alpha'_2 = Aw$ and $\alpha_2 = tw$ for some rule $A \rightarrow t \in R$
- Hence $S \xrightarrow{*} \alpha_1\alpha_2$

PDA $\implies$ CFL

Lemma 19

$\mathcal{L}_{\mathsf{PDA}} \subseteq \mathcal{L}_{\mathsf{CFG}}$*: If a PDA accepts a language then it is context free.*

We prove the lemma by constructing a CFG G for a language $\mathcal{L}$ accepted by a PDA P

Let $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$. We assume wlg. that:

- A single accepting state $q_a \in F$.
- P empties the stack before accepting
- Each transition either pops or pushes

Can we justify the above?

Proof Idea

- Suppose string x takes P from state p with empty stack to state q with empty stack:
- First move that touches the stack must be a push, last must be a pop.
- In between:
- *Either stack is empty only at start and finish*:
- Simulate by $A_{pq} \to aA_{rs}b$, where a, b are first and last symbols in x, r is state that p can reach in a step and s is state that can reach q in a step.
- *or stack was empty at some point in between*:
- Simulate by $A_{pq} \to A_{pr}A_{rq}$ where r is intermediate state and P has empty stack.

Defining $G = (V, \Sigma, R, S)$

- $V = \{A_{pq} : p, q \in Q\}$

  Idea: A_{pq} will generate all strings that take P from p with an empty stack, to q with an empty stack

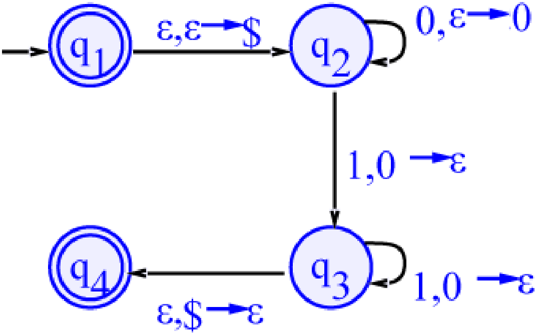
- $S = A_{q_0, q_a}$
- Initially $R = \emptyset$ and
  1. Add $\{A_{pq} \to A_{p,r}A_{r,q} : p, q, r \in Q\}$ to R
  2. Add $\{A_{qq} \to \varepsilon : q \in Q\}$ to R
  3. For all $p, r, s, q \in Q$, $a, b \in \Sigma_\varepsilon$ and $t \in \Gamma$ such that

     3.1 $(r, t) \in \delta(p, a, \varepsilon)$ and

     3.2 $(q, \varepsilon) \in \delta(s, b, t)$

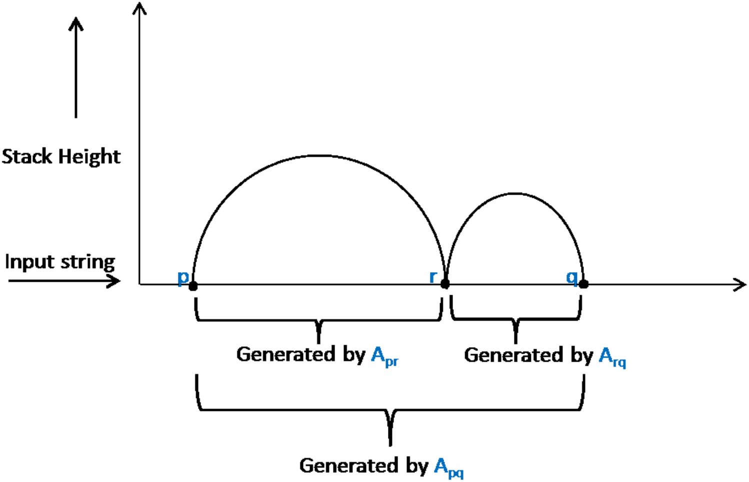
     add $A_{pq} \to aA_{r,s}b$ to R

Example PDA to CFG

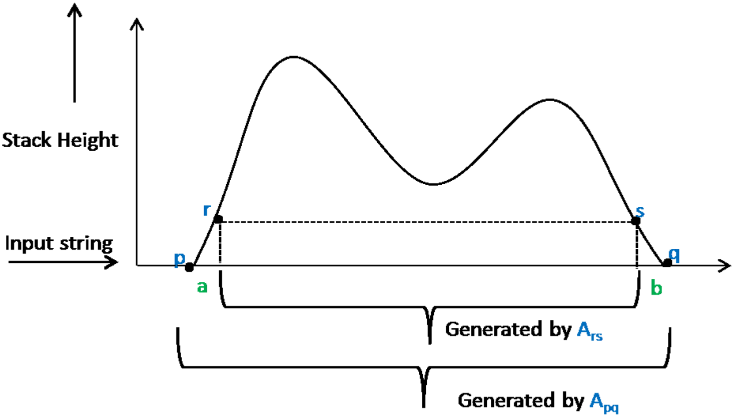


$A_{q_1,q_4} \to A_{q_2,q_3}$
$A_{q_2,q_3} \to 0A_{q_2,q_3}1$
$A_{q_2,q_3} \to 0A_{q_2,q_2}1$.
$A_{q_2,q_2} \to \varepsilon$.

PDA Computation corresponding to $A_{pq} \to A_{p,r}A_{r,q}$



PDA Computation corresponding to $A_{pq} \to aA_{r,s}b$



Claim: $\mathcal{L}(G) = \mathcal{L}(P)$

Claim 20

$A_{pq} \overset{*}{\to} w \in \Sigma^*$ iff $(q, \varepsilon) \in \widehat{\delta}(p, w, \varepsilon)$

Proof by induction on the number of derivation rules/ transitions

A Short Summary

- Regular Languages $\equiv$ Finite Automata.
- Context Free Languages $\equiv$ Push Down Automata.
- Closure properties of regular languages and of CFLs.
- Most algorithmic problems for finite automata are solvable.
- Some algorithmic problems for finite automata are not solvable.
- Pumping lemmata for both classes of languages.
- There are additional languages out there.

The View Over The Horizon