Computational Models - Exercise #3, Spring 2014/15

Due: May 13

Note: Question 7 is optional and a (correct) solution will grant a modest bonus.

1. Let $G = (\{S, A\}, \{a, b, c\}, R, S)$ be a CFG where R contains the rules:

$$\begin{aligned} S &\rightarrow abScA \mid \varepsilon \\ A &\rightarrow bA \mid b \end{aligned}$$

   What is $L(G)$? No need for a formal proof, but do provide an explanation.

2. For each of the following languages over $\Sigma = \{a, b, c\}$, present a diagram representing a PDA (no need for a correctness proof).

   (a) $L_1 = \left\{a^i b^j a^i \mid i, j \geq 0\right\}$. Write a formal definition of the PDA as well.
   (b) $L_2 = \left\{a^i b^j \mid j \geq 0, i \leq j \leq 2i\right\}$.
   (c) $L_3 = \left\{a^i b^j c^k \mid i, j, k \geq 0,\ |i - k| = j\right\}$.
   (d) $L_4 = \{w \in \Sigma^\star \mid w \text{ does not equal } xcx \text{ for some } x \in \{a, b\}^\star\}$.

3. For each of the following languages, present a formal definition of a CFG (no need for a correctness proof, but do provide an explanation).

   (a) $L_5 = \{w \in \Sigma^\star \mid \#_a(w) = \#_b(w)\}$ over $\Sigma = \{a, b\}$. Also, show a derivation tree for the word $abbbaa$.
   (b) $L_6 = \{x\$y \mid x, y \in \{a, b\}^\star \wedge |x| \neq |y|\}$ over $\Sigma = \{a, b, \$\}$. Also, show a derivation tree for the word $abbb\$aa$.

4. Prove using the Pumping Lemma that $L_7 = \{a^n b^m c^n d^m \mid n, m \geq 0\}$ over $\Sigma = \{a, b, c, d\}$ is not context-free.

5. Let L_r be any regular language and L_c any CFL, both over some alphabet Σ. Formally prove/disprove:

   (a) $L' = \{u\sigma v \mid uv \in L_r \wedge \sigma \in \Sigma \wedge |u| = |v|\}$ is context-free.
   (b) $L'' = \left\{uv \mid u \in L_c \wedge v \in L_c^R \wedge |u| = |v|\right\}$ is context-free[1].

6. Note that an algorithm is a process that <u>halts</u> on every input and returns the correct answer.

   (a) Let G be a CFG in Chomsky Normal Form with n variables. Prove that $L(G)$ is infinite if and only if there exists $w \in L(G)$ such that $2^n < |w| \leq 2^{n+1}$.

[1] Recall that L^R is the reverse of L. That is, $L^R = \left\{w^R \mid w \in L\right\}$ and $(w_1 \cdots w_n)^R = w_n \cdots w_1$.

(b) Show an algorithm that given a CFG G with n variables decides whether $L(G)$ is infinite.

(c) Show an algorithm that given a CFG G with n variables decides whether $|L(G)| = 2015$.

7.* Let L be a language over $\Sigma = \{a, b\}$. Define

$$Minus(L) = \left\{a^i b^j \mid \exists w \in L.|w| = i - j\right\}.$$

Prove that if L is regular then $Minus(L)$ is context-free[2].

[2]Given two languages A and B over Σ, recall that $A/B = \{x \in \Sigma^\star \mid \exists y \in B.xy \in A\}$. You can use, without proof, the fact that if A is a CFL and B is a regular language then A/B is a CFL.