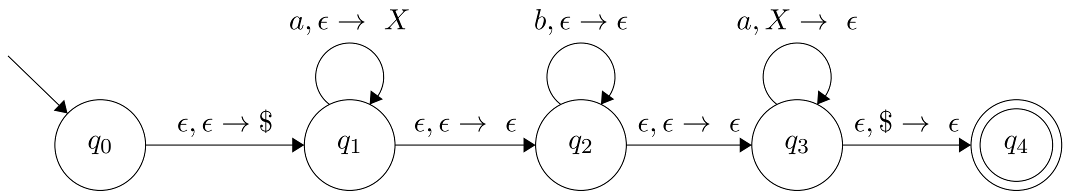
Computational Models - Exercise #3 solution sketch

1. $L = \left\{(ab)^n(cw)^n \mid n \geq 0, w \in \left\{b^i \mid i > 0\right\}\right\}$.

2. (a) The PDA for L_1:

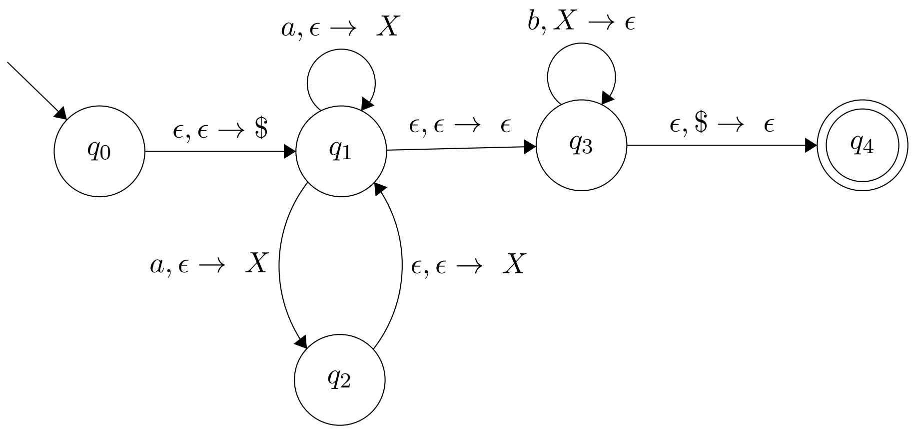


A formal definition: $P = (\{q_0, q_1, q_2, q_3, q_4\}, \{a, b, \epsilon\}, \{\$, X\}, \delta, q_0, \{q_4\})$, where:

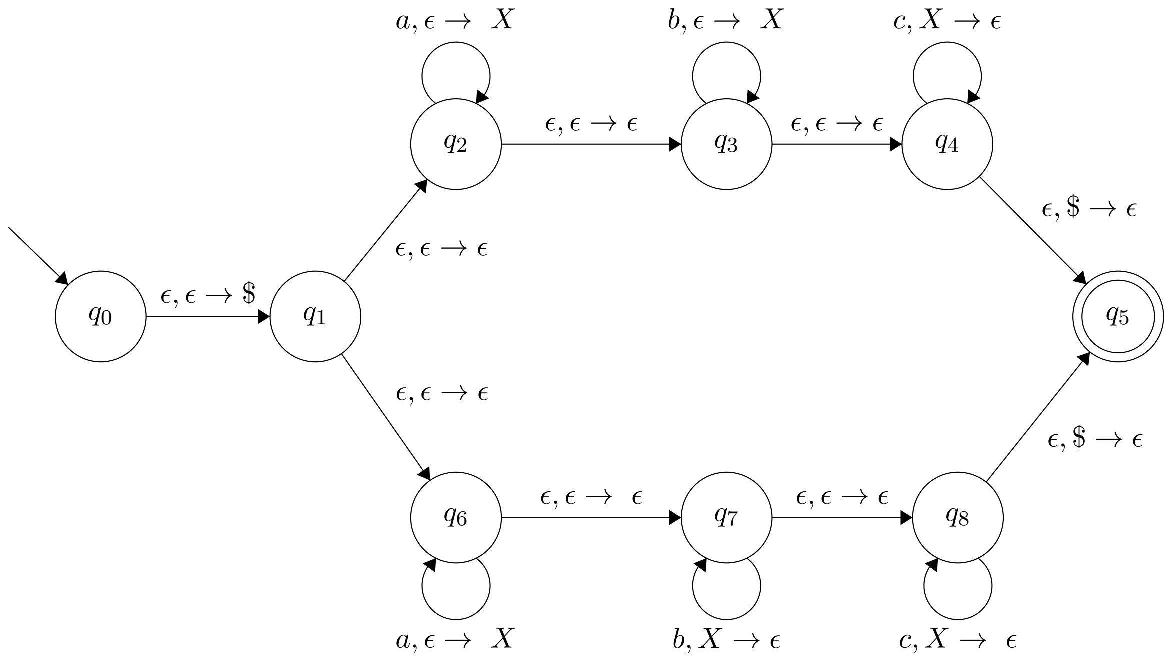
$$\begin{aligned}
\delta(q_0, \epsilon, \epsilon) &= \{(q_1, \$)\} \\
\delta(q_1, a, \epsilon) &= \{(q_1, X)\} \\
\delta(q_1, \epsilon, \epsilon) &= \{(q_2, \epsilon)\} \\
\delta(q_2, b, \epsilon) &= \{(q_2, \epsilon)\} \\
\delta(q_2, \epsilon, \epsilon) &= \{(q_3, \epsilon)\} \\
\delta(q_3, a, X) &= \{(q_3, \epsilon)\} \\
\delta(q_3, \epsilon, \$) &= \{(q_4, \epsilon)\},
\end{aligned}$$

and $\emptyset$ elsewhere.

(b) The PDA for L_2:



(c) The PDA for L_3:



(d)* L_4 can be written as $L_4 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, where:

- $A_1 = \{w \in \Sigma^\star \mid \#_c(w) \neq 1\}$.
- $A_2 = \{x_1 c x_2 \mid x_1, x_2 \in \{a,b\}^\star \wedge |x_1| < |x_2|\}$.
- $A_3 = \{x_1 c x_2 \mid x_1, x_2 \in \{a,b\}^\star \wedge |x_1| > |x_2|\}$.
- $A_4 = \{x_1 a y_1 c x_2 b y_2 \mid x_1, x_2, y_1, y_2 \in \{a,b\}^\star \wedge |x_1| = |x_2|\}$.
- $A_5 = \{x_1 b y_1 c x_2 a y_2 \mid x_1, x_2, y_1, y_2 \in \{a,b\}^\star \wedge |x_1| = |x_2|\}$.

A PDA for every A_i itself is not hard, and the overall PDA for L_4 is attained by a simple union.

3. (a) $G_5 = (\{S\}, \{a,b\}, R, S)$ where R has the single rule:

$$S \to SaSbS \mid SbSaS \mid \varepsilon.$$

(b) $G_6 = (\{S, L, R, A, B\}, \{a, b, \$\}, \tilde{R}, S)$ where $\tilde{R}$ has the rules:

$$\begin{aligned} S &\to BL \mid RB \\ L &\to BL \mid A \\ R &\to RB \mid A \\ A &\to BAB \mid \$ \\ B &\to a \mid b \end{aligned}$$

4. Assume to the contrary that L_7 is context-free and let $\ell > 0$ be the promised pumping constant. Take $s = a^\ell b^\ell c^\ell d^\ell \in L_7$ and obviously $|s| \geq \ell$. Consider a general decomposition $s = uvxyz$ such that $|vy| > 0$ and $|vxy| \leq \ell$. We shall see that in any such decomposition, $w = uv^0xy^0z \notin L_7$, in contradiction to the Pumping Lemma. In every such decomposition vxy satisfies one of the following:

(a) vxy contains only a-s, so $vy = a^s$ for $s > 0$.
(b) vxy contains only b-s, so $vy = b^s$ for $s > 0$.
(c) vxy contains only c-s, so $vy = c^s$ for $s > 0$.
(d) vxy contains only d-s, so $vy = d^s$ for $s > 0$.
(e) vxy contains only a-s and b-s, so $vy = a^s b^t$ for $s + t > 0$.
(f) vxy contains only b-s and c-s, so $vy = b^s c^t$ for $s + t > 0$.
(g) vxy contains only c-s and d-s, so $vy = c^s d^t$ for $s + t > 0$.

In form (1) (and likewise for the following three), $w = a^{\ell-s}b^\ell c^\ell d^\ell$. As $s > 0$, $w \notin L_7$. In form (5) (and likewise for the following two), $w = a^{\ell-s}b^{\ell-t}c^\ell d^\ell$. As $s + t \geq 1$ then either $s \geq 1$ or $t \geq 1$, so either $\#_a(w) = \ell - s < \ell = \#_c(w)$ or $\#_b(w) = \ell - t < \ell = \#_d(w)$. In any case, $w \notin L_7$.

5. (a) The claim is correct. Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA that accepts L_r. The idea is to construct a PDA that consists of two copies of M. The first copy pushes a symbol X to the stack after every character read and the second pops. We can always "guess" which σ takes us from the first copy to the second. We accept iff we are in an accepting state of M in the second copy and the stack is empty. Formally, construct a PDA

$$M' = (Q \times \{1, 2\} \cup \{q_i, q_a\}, \Sigma, \{\perp, X\}, \delta', q_i, \{q_a\})$$

where:

- $\delta'(q_i, \varepsilon, \varepsilon) = \{((q_0, 1), \perp)\}$.
- $\forall q \in Q \forall \sigma \in \Sigma . \delta'((q, 1), \sigma, \varepsilon) = \{((\delta(q, \sigma), 1), X), ((q, 2), \varepsilon)\}$.
- $\forall q \in Q \forall \sigma \in \Sigma . \delta'((q, 2), \sigma, X) = \{((\delta(q, \sigma), 2), \varepsilon)\}$.
- $\forall q \in F . \delta'((q, 2), \varepsilon, \perp) = \{(q_a, \varepsilon)\}$.

For correctness we give the first direction (the second direction is almost identical):

$$\begin{aligned}
x \in L' &\Rightarrow \exists u, \sigma, v . uv \in L_r \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v . uv \in L(M) \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' . \hat{\delta}(q_0, u) = q \wedge \hat{\delta}(q, v) = q' \wedge q' \in F \wedge \sigma \in \Sigma \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' . ((q_0, 1), \perp) \in \delta'(q_i, \epsilon, \epsilon) \wedge ((q, 1), \perp X^{|u|}) \in \hat{\delta}'((q_0, 1), u, \perp) \wedge \\
&\qquad ((q, 2), \epsilon) \in \delta'((q, 1), \sigma, \epsilon) \wedge ((q', 2), \perp X^{|u|-|v|}) \in \hat{\delta}'((q, 2), v, \perp X^{|u|}) \wedge q' \in F \wedge |u| = |v| \\
&\Rightarrow \exists u, \sigma, v, q, q' . ((q_0, 1), \perp) \in \delta'(q_i, \epsilon, \epsilon) \wedge ((q, 1), \perp X^{|u|}) \in \hat{\delta}'((q_0, 1), u, \perp) \wedge \\
&\qquad ((q, 2), \epsilon) \in \delta'((q, 1), \sigma, \epsilon) \wedge ((q', 2), \perp) \in \hat{\delta}'((q, 2), v, \perp X^{|u|}) \wedge (q_a, \epsilon) \in \delta'((q, 2), \epsilon, \perp) \\
&\Rightarrow \exists u, v . (q_a, \epsilon) \in \hat{\delta}'(q_i, uv, \epsilon) \\
&\Rightarrow \exists x \in L(M').
\end{aligned}$$

Note that the claims $((q, 1), \perp X^{|u|}) \in \hat{\delta}'((q_0, 1), u, \perp)$ and $((q', 2), \perp X^{|u|-|v|}) \in \hat{\delta}'((q, 2), v, \perp X^{|u|})$ require an induction on the length of u and v.

(b) The claim is incorrect. Let $L_c = \{a^n b^n \mid n \geq 0\}$. Then:

$$L_1 = \{uv \mid u \in L_c \wedge v \in L_c \wedge |u| = |v|\} = \left\{a^n b^{2n} a^n \mid n \geq 0\right\}.$$

We will prove that L_1 is not context-free. Assume to the contrary, and define $h_1 : \{a,b,c\} \to \{b,c\}^\star$ as $h_1(a) = h_1(c) = c$ and $h_1(b) = b$. Then,

$$L_2 = h_1^{-1}(h_1(L_1)) \cap L[a^\star b^\star c^\star] = \left\{a^n b^{2n} c^n \mid n \geq 0\right\}$$

is also context-free. Define $h_2 : \{a,b,b',c\} \to \{a,b',c\}^\star$ as $h_2(b) = h_2(b') = b'$, $h_2(a) = a$ and $h_2(c) = c$. Then,

$$L_3 = h_2^{-1}(h_2(L_2)) \cap L[a^\star (bb')^\star c^\star] = \left\{a^n (bb')^n c^n \mid n \geq 0\right\}$$

is also context-free. Finally, define $h_3 : \{a,b,b',c\} \to \{a,b,c\}$ as $h(a) = a$, $h(b) = b$, $h(c) = c$ and $h(b') = \varepsilon$. We then have that

$$L_4 = h_3(L_3) = \{a^n b^n c^n \mid n \geq 0\}$$

is context-free, in contradiction to what we saw in class.

6. Note that an algorithm is a process that halts on every input and returns the correct answer.

   (a) If there exists a $w \in L(G)$ such that $|w| > 2^n$ then from the Pumping Lemma for context-free languages, w can be pumped for infinitely many i-s and $L(G)$ is infinite.
   If $L(G)$ is infinite there exists a word $w \in L(G)$ such that $|w| > 2^n$. The parse tree of minimal size for w contains a variable A that appears twice on some path from the root to some leaf. We will remove all duplicate variables on the same path by shrinking, as in the Pumping Lemma, and get a word w' such that $|w'| \leq 2^n$. Then we start pumping A. Assuming we chose A to be the lowest variable that appear twice on some path, it is guaranteed that $|vy| \leq 2^n$ and we will eventually get a word in $L(G)$ of the proper length.

   (b) By the previous section, given a CFG G we can check if G generates any word w (go over all possible derivations), such that $2^n < |w| \leq 2^{n+1}$. If one of the words is generated by G, then $L(G)$ is infinite, otherwise – finite.

   (c) First we check if $L(G)$ is infinite. If it is, we return *false*. Otherwise, we enumerate over all words of length at most 2^n and count how many are accepted. We return *true* iff the count is 2015.

7.* Define the homomorphism $h : \{a,b\} \to \{b\}^\star$ as $h(a) = h(b) = b$. Then,

$$h(L) = \left\{b^k \mid \exists w \in L. |w| = k\right\}.$$

Note that:

$$\begin{aligned}
\{a^n b^n \mid n \geq 0\} / h(L) &= \{x \in \{a,b\}^\star \mid \exists y \in h(L) \wedge xy \in \{a^n b^n \mid n \geq 0\}\} \\
&= \left\{a^i b^j \mid \exists b^k \in h(L) \wedge a^i b^{j+k} \in \{a^n b^n \mid n \geq 0\}\right\} \\
&= \left\{a^i b^j \mid \exists b^k \in h(L) \wedge i = j + k\right\} \\
&= \left\{a^i b^j \mid \exists w \in L \wedge |w| = k = i - j\right\} \\
&= Minus(L).
\end{aligned}$$

L is regular, so $h(L)$ is regular as well. $\{a^n b^n \mid n \geq 0\}$ is known to be context-free, so $\{a^n b^n \mid n \geq 0\} / h(L) = Minus(L)$ is context-free as well.