Computational Models - Exercise #6, Spring 2014/15

Due: June 21

Note: We say a language L is nontrivial if $L \neq \emptyset$ and $L \neq \Sigma^\star$. We say a property $C \subseteq \mathcal{RE}$ is nontrivial if $C \neq \emptyset$ and $C \neq \mathcal{RE}$. You can assume $|\Sigma| \geq 2$. Question 3(c) is a bonus question.

1. Let $L \in \mathcal{NPC}$. Prove that $L_p = \{\langle x, y\rangle \mid x \in L \wedge y \in \Sigma^\star\} \in \mathcal{NPC}$.

2. Prove/disprove:

   (a) For every two nontrivial languages $A, B \in \mathcal{R}$, $A \leq_m B$.

   (b) For every two nontrivial languages $A, B \in \mathcal{RE}$, $A \leq_m B$.

   (c) For every two nontrivial properties $C_1, C_2 \subseteq \mathcal{RE}$, $L_{C_1} \leq_m L_{C_2}$.

   (d) $L_1 = \{\langle M\rangle \mid \text{every string accepted by } M \text{ has an equal number of 0-s and 1-s}\} \in \mathcal{R}$.

   (e) $L_2 = \{\langle M\rangle \mid |\Sigma^\star \setminus L(M)| \leq 2\} \notin \mathcal{RE} \cup \text{co}-\mathcal{RE}$ (In words: There are at most 2 words that are not in $L(M)$).

3. Prove:

   (a) If $L \in \mathcal{P}$ then $L^\star \in \mathcal{P}$.

   (b) If $L \in \mathcal{NP}$ then $L^\star \in \mathcal{NP}$.

   (c)* If $L \in \text{co}-\mathcal{NP}$ then $L^\star \in \text{co}-\mathcal{NP}$.

4. We say that a polynomial reduction f is a *shrinking reduction* if there exists n_0 such that for every $x \in \Sigma^\star$ such that $|x| \geq n_0$, $|f(x)| \leq |x| - 1$. Assuming $\mathcal{P} \neq \mathcal{NP}$, prove/disprove:

   (a) For every two nontrivial languages $A, B \in \mathcal{P}$ there exists a shrinking reduction from A to B.

   (b) For every two nontrivial languages $A, B \in \mathcal{NPC}$ there exists a shrinking reduction from A to B.

5. We say that two languages A and B are *polynomially equivalent* if $A \leq_p B$ and $B \leq_p A$. Prove/disprove:

   (a) VC[1], CLIQUE and IS are polynomially equivalent[2].

   (b) H_{TM} and $\overline{H_{TM}}$ are polynomially equivalent.

[1] A vertex cover of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $\{u, v\} \in E$ then $u \in V'$ or $v \in V'$ (or both). The language VC is the language of all $(\langle G\rangle, k)$ such that G has a vertex cover of size k.

[2] You *cannot* use the fact that all three problems are in $\mathcal{NPC}$, but you *can* state and prove the transitivity of polynomial reductions.

(c) Assuming $\mathcal{P} \neq \mathcal{NP}$, every two nontrivial languages in $\mathcal{NP}$ are polynomially equivalent.

6. A Hamiltonian path is a path in a graph that visits each vertex exactly once. A Hamiltonian cycle in a graph is a Hamiltonian path that is a cycle. Let

$$\mathsf{HamPath} = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a Hamiltonian path from } s \text{ to } t\}$$

and $\mathsf{HamCycle} = \{\langle G\rangle \mid G$ is a directed graph that has a Hamiltonian cycle$\}$.

   (a) Prove: $\mathsf{HamCycle} \leq_p \mathsf{HamPath}$ (you need to explicitly show the reduction and prove its correctness).

   (b) Let $\mathsf{UnHamPath}$ be the undirected variant of $\mathsf{HamPath}$. Use the fact that $\mathsf{HamPath}$ is $\mathcal{NP}$-complete to show that $\mathsf{UnHamPath} \in \mathcal{NPC}$.

7. Let $\mathsf{UpToOneSAT} = \{\langle\varphi\rangle \mid \varphi$ is a 3CNF formula that has at most one satisfying assignment$\}$. Prove that $\mathsf{UpToOneSAT} \in \mathcal{NP}$ implies $\mathcal{NP} = \text{co}-\mathcal{NP}$.