Solution sketch 4 - Computational Models - Spring 2015

1. Prove or disprove:

   (a) False. We know A_{TM} in $\mathcal{RE}$, but $\overline{A_{TM}}$ isn't.

   (b) True. Given $L_1, L_2 \in \mathcal{RE}$ we can construct a TM that accepts $L_1 \cap L_2$. Let M_1 be TM that accepts L_1 and M_2 a TM that accepts L_2, we can simply run M_1 on our input, then run M_2 on our input, and accept iff both accepted.

   (c) True. Use definitions, De-Morgan rule and the fact that $\mathcal{RE}$ is closed under union.

2. (a) Use a two tape TM where the second tape will simulate the stack.
   $M = (Q', \Sigma, \Gamma, \delta', q_0, q_a, q_r)$ where:
   $Q' = Q \cup \{q_a, q_r\}$, $\Gamma = \Sigma \cup \{\sqcup\}$
   $\delta'(q, \sigma) = (\delta(q, \sigma), \sigma, R)$ for all $\sigma \in \Sigma$
   $\delta'(q, \sqcup) = (q_a, \sqcup, R)$ if $q \in F$
   $\delta'(q, \sqcup) = (q_r, \sqcup, R)$ if $q \notin F$

   (b) $M = (Q', \Sigma, \Gamma, \delta', q_0, q_a, q_r)$ where:
   $Q' = Q \cup \{q_a, q_r\} \cup Q_{new}$, $\Gamma = \Sigma \cup \{\sqcup\}$
   The transition function is $\delta' : Q' \times \Gamma^2 \mapsto Q' \times \Gamma^2 \times \{L, R, S\}^2$ such that the head of the first tape always moves right and we always keep the context of the first tape intact.

   - Reading a blank on the the second tape is like having an empty stack so we cannot perform a pop operation. We can follow all operation which include no pops. The second head stays in place:
     $\delta'(q, \sigma_1, \sqcup) = (q', \sigma_1, \sigma_2', R, S)$ such that $\delta(q, \sigma_1, \varepsilon) = (q', \sigma_2')$
     Now, we know that the second head points to the top of the simulated stack.

- For every pop operation with no push operation we need to move the second head left. Thus, for every transition of the form $\delta(q, \sigma_1, \sigma_2) = (q', \varepsilon)$ add a transition $\delta'(q, \sigma_1, \sigma_2) = (q', \sigma_1, \sqcup, R, L)$
- For every pop operation with a push operation we need to replace the head of the stack and not move the second head: Thus, for every transition of the form $\delta(q, \sigma_1, \sigma_2) = (q', \sigma_2')$ add a transition $\delta'(q, \sigma_1, \sigma_2) = (q', \sigma_1, \sigma_2', R, S)$
- For every push operation with no pop operation we need to move the head right and then write on the top of the stack. We do this by adding an intermediate transition that remembers our state and what we want to write on the second tape. I.e. we add the following states: $Q_{new} = \{(q, \sigma) \text{ s.t. } q \in Q \text{ and } \sigma \in \Sigma\}$ and the following generic rule (for all $\sigma_1 \in \Sigma$ and all new states): $\delta'((q, \sigma_2), \sigma_1, \sqcup) = (q, \sigma_1, \sigma_2, S, S)$. Now, for every transition of the form $\delta(q, \sigma_1, \varepsilon) = (q', \sigma_2')$ add the following transition $\delta'(q, \sigma_1, \sigma_2) = ((q', \sigma_2'), \sigma_1, \sigma_2, R, R)$
- When we finished reading the input we check if we are in an accepting state or not:
  $\delta'(q, \sqcup, \sqcup) = \delta'(q, \sqcup, \sigma) = (q_a, \sqcup, \sqcup, S, S)$ if $q \in F$
  $\delta'(q, \sqcup, \sqcup) = \delta'(q, \sqcup, \sigma) = (q_r, \sqcup, \sqcup, S, S)$ if $q \notin F$

3. Let $L \in RE$. Let M_L be a TM that accepts L. That is $L(M_L) = L$. Now, a TM that accepts Prefix(L) will do the following: On input x it will use a monotone enumerator for Σ^* to concurrently simulate M_L on all words of the form xy for $y \in \Sigma^*$ (as was shown in the recitation for the TM that accepts $\overline{E_{TM}}$), and will accept if M_L accepts any of the inputs.

4. Let M_L be a TM that accepts $L_f \in RE$. A TM that computes f is the following: On input x, it simulates (simultaneously, using an enumerator for Γ^*) the computations of M_L on all inputs (x, y), and outputs y whenever M_L accepts (x, y). For the other direction, let M be a TM that computes f. A TM that accepts L_f is the following: on input (x, y) it simulates the computation of M on input x and accepts if the result of the computation equals y.

5. (a) False. Consider any undecidable L, and $L_1 = \phi$, $L_2 = \Sigma^*$.
   (b) False.
   Let $L_1 = \{\langle M, x\rangle \mid M \text{ is a TM that halts on x and has an even number of states}\}$
   and $L_2 = \{\langle M, x\rangle \mid M \text{ is a TM that halts on x and has an odd number of states}\}$
   It is easy to see that $L_1, L_2 \in \mathcal{RE} \setminus \mathcal{R}$ but their intersection is empty hence in $\mathcal{R}$.

(c) False.
Let $L_1 = \{\langle M, x\rangle \mid M$ is a TM that halts on x or has an even number of states$\}$
and $L_2 = \{\langle M, x\rangle \mid M$ is a TM that halts on x or has an odd number of states$\}$
It is easy to see that $L_1, L_2 \in \mathcal{RE} \setminus \mathcal{R}$ but their union is all pairs $\langle M, x\rangle$ hence in $\mathcal{R}$.

(d) True. Otherwise we could construct a TM that decides L_1: Let M_1, M_2 be the TM that accepts L_1 and L_2, respectively, and let M_u, M_i be the TM that decides $L_1 \cup L_2$ and $L_1 \cap L_2$, respectively. A TM M that decides L_1 on input x:

- Run M_i on x. If M_i accepts, M accepts
- Run M_u on x. If M_u rejects, M rejects
- Run M_1, M_2 simultaneously on x until one accepts
  - If M_1 accepts, M accepts
  - If M_2 accepts, M rejects